

Gauging without Initial Symmetry

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The gauge principle is at the heart of a good part of fundamental physics: Starting with a group G of so-called rigid symmetries of a functional defined over space-time Σ , the original functional is extended appropriately by additional $\text{Lie}(G)$ -valued 1-form gauge fields so as to lift the symmetry to $\text{Maps}(\Sigma, G)$. Physically relevant quantities are then to be obtained as the quotient of the solutions to the Euler-Lagrange equations by these gauge symmetries.

In this article we show that one can construct a gauge theory for a standard sigma model in arbitrary space-time dimensions where the target metric is not invariant with respect to *any* rigid symmetry group, but satisfies a much weaker condition: It is sufficient to find a collection of vector fields v_a on the target M satisfying the *extended Killing equation* $v_{a(i;j)} = 0$ for *some* connection acting on the index a . For regular foliations this is equivalent to merely requiring the distribution orthogonal to the leaves to be invariant with respect to leaf-preserving diffeomorphisms of M .

The resulting gauge theory has the usual quotient effect with respect to the original ungauged theory: in this way, much more general orbits can be factored out than usually considered. In some cases these are orbits that do not correspond to an initial symmetry, but still can be generated by a finite dimensional Lie group G . Then the presented gauging procedure leads to an ordinary gauge theory with Lie algebra valued 1-form gauge fields, but showing an unconventional transformation law. In general, however, one finds that the notion of an ordinary structural Lie group is too restrictive and should be replaced by the much more general notion of a structural Lie groupoid.

INTRODUCTION

The Standard Model of elementary particle physics, but also General Relativity and String Theory, are gauge theories. In the former case, for example, gauging of an $\text{SU}(3)$ rigid symmetry rotation between the three quarks leads to the introduction of the eight gluons that mediate the interaction between those elementary particles. Mathematically the resulting theory is described by connections in a principle bundle (in the above example with the structure group $\text{SU}(3)$, the connection 1-forms representing the $\dim \text{SU}(3) = 8$ gluons) with the matter fields being sections in appropriate associated vector bundles.

The procedure can be generalized to bundles where the matter fields are sections in arbitrary fiber bundles, the fibers being equipped with appropriate geometric structures that are invariant w.r.t. some group G , the “rigid” symmetry group. In the case of a trivial bundle, sections can be identified with maps from the base manifold Σ to the fiber M and one obtains a sigma model, such as e.g. the “standard” one:

$$S_0[X] = \int_{\Sigma} \frac{1}{2} g_{ij}(X) dX^i \wedge *dX^j. \quad (1)$$

This is a functional on smooth maps $X: \Sigma \rightarrow M$, where the d -dimensional spacetime Σ as well as the n -dimensional target manifold M are equipped with a

(possibly Lorentzian signature) metric h and g , respectively. The metric h enters the functional (1) implicitly by means of its induced Hodge duality operation $*$, which becomes more transparent if one rewrites the integrand of (1) in components: $\sqrt{|\det(h)|} h^{\mu\nu} g_{ij}(X) \partial_{\mu} X^i \partial_{\nu} X^j d^d \sigma$.

Symmetries of the geometrical data on the source manifold Σ or on the target manifold M lift to symmetries of any functional that uses only such data on the map space between these two manifolds. So, in the case of (1) an invariance of h and g lead to an invariance of S_0 . By the Noether procedure this gives rise to conserved quantities then. For example, if h is a flat metric on Σ , the appropriate conserved quantities correspond to the energy momentum tensor $T^{\mu\nu}$ of the theory.

Let us, on the other hand, suppose that the metric g has a nontrivial isometry group G , which infinitesimally implies $\mathcal{L}_v g = 0$, valid for the vector fields $v = \rho(\xi)$ on M corresponding to arbitrary elements $\xi \in \mathfrak{g} = \text{Lie}(G)$, ρ denoting the representation of \mathfrak{g} on M induced by the G -action. In this case, there is a canonical procedure to lift the induced G -symmetry of S_0 to a gauge symmetry on an extended functional S_1 . This procedure is called *minimal coupling*: After introducing \mathfrak{g} -valued 1-forms $A = A^a \xi_a \in \Omega^1(M, \mathfrak{g})$, ξ_a denoting any basis of \mathfrak{g} , ordinary derivatives dX^i on the scalar fields X^i , which are local functions on Σ , are replaced by covariant ones,

$$DX^i := dX^i - \rho_a^i(X) A^a, \quad (2)$$

where $\rho_a^i(x)\partial_i \equiv \rho(\xi_a)$. The new functional

$$S[X, A] = \int_{\Sigma} \frac{1}{2} g_{ij}(X) DX^i \wedge *DX^j \quad (3)$$

is now invariant with respect to the combined infinitesimal gauge symmetries generated by

$$\delta X^i = \rho_a^i(X) \varepsilon^a, \quad (4)$$

$$\delta A^a = d\varepsilon^a + C_{bc}^a A^b \varepsilon^c, \quad (5)$$

for arbitrary $\varepsilon^a \in C^\infty(\Sigma)$. Here C_{bc}^a are the structure constants of the Lie algebra \mathfrak{g} in the chosen basis.

In the space of (pseudo) Riemannian metrics, those permitting a non-trivial invariance or isometry group G are the big exception. A generic metric g does not permit *any* non-vanishing vector field v such that Killing's equation $\mathcal{L}_v g = 0$ holds true. Also in general there is also a relatively small upper bound to the number of independent Killing vector fields, one has $\dim G \leq n(n+1)/2$ where $n \equiv \dim M$.

It is conventional belief that a non-trivial isometry group G is necessary to gauge the action functional (1). It is the purpose of this letter to show that this is far from true. Our observation will have two parts: The first one is that there may be *group* actions on a Riemannian manifold M used as a target of a sigma model that are not isometries but still can be gauged by more or less standard methods (in particular the introduction of 1-forms gauge fields taking values in the Lie algebra of the group acting on M). Second, and maybe more important, one does not need to restrict to the action of a finite dimensional Lie group in general. Instead, it is sufficient to have the action of a Lie groupoid \mathcal{G} on M . In fact, the use of Lie groupoids (and their associated Lie algebroids) in the context of gauge theories is even *suggested* by the present analysis as the much more generic one.[1]

THE CASE OF 1-DIMENSIONAL LEAVES

For a conceptual orientation, we first consider the highly simplified situation of a (regular) foliation of M into *one-dimensional*, hyper-surface-orthogonal leaves for a positive definite metric g . In this case we can choose an adapted local coordinate system such that ∂_1 generates these leaves and the hyper-surfaces $x^1 = \text{const}$ are orthogonal to them everywhere. This implies that $g_{1i} = 0$ for all $i \neq 1$ or, if we denote those indices by Greek letters from the beginning of the alphabet, that $g_{1\alpha} = 0$ (while certainly $g_{11} > 0$). ∂_1 *not* generating an isometry is tantamount to $g_{ij,1} \neq 0$, for at least some components.

According to standard folklore, it should not be possible to extend S_0 by gauge fields such that ∂_1 becomes a direction of gauge symmetries, i.e. such that

$$\delta X^1 = \varepsilon \quad (6)$$

will leave the extended action gauge invariant for an arbitrary choice of the parameter function $\varepsilon \in C^\infty(\Sigma)$ (together with an appropriate transformation of the gauge field certainly); we will see in this example, what is really necessary for a promotion of the leaf to a local symmetry direction—although, by assumption, these leaves are *not* orbits of a rigid symmetry group G (we rather assume that the group G of isometries of g is trivial, $G = \{\text{id}\}$).

Since the leaves are 1-dimensional, we will introduce also only one gauge field $A \in \Omega^1(\Sigma)$ and consider the action functional

$$S[X, A] = \int_{\Sigma} \frac{1}{2} g_{11}(X) (dX^1 - A) \wedge *(dX^1 - A) + \frac{1}{2} g_{\alpha\beta}(X) dX^\alpha \wedge *dX^\beta. \quad (7)$$

If we postulate the conventional $\delta A = d\varepsilon$, we achieve that $dX^1 - A$ is strictly gauge invariant. It is then easy to see that with this transformation of the gauge field, we necessarily need $g_{ij,1} = 0$, which would imply that ∂_1 generates isometries of g . However, one notices that changes of g_{11} under the flow of ∂_1 can be compensated by means of a modified transformation of the gauge field A . It is thus sufficient to require merely

$$g_{\alpha\beta,1} = 0 \quad (8)$$

for gauge invariance of (7) if the transformation (6) is amended by

$$\delta A = d\varepsilon + \frac{\varepsilon}{2} (\ln(g_{11}))_{,1} (dX^1 - A). \quad (9)$$

This makes sense also geometrically: If we want to factor out the one-dimensional leaves equipped with the metric g_{11} , it is not necessary that this metric, that disappears after all in the quotient, is invariant along the leaves. Instead, what is needed only is that the transversal metric $g_{\alpha\beta}$ is invariant along these leaves, as expressed by Equation (8).

A conceptual understanding and practical generalization of this idea is obvious: Consider the Cartesian product $M = M_1 \times M_2$ of two Riemannian manifolds $(M_1, g_{(1)})$ and $(M_2, g_{(2)})$, which becomes Riemannian itself by means of $g = g_{(1)} + g_{(2)}$, where certainly one needs to take the pullback of the 2-tensor on each of the two factors by the respective projection map. If one wants to construct a sigma model with target M_2 in terms of a “quotient construction” using the idea of gauge theories, where one starts with a target that is all of M , it should not play any role that the total metric g is not invariant on $M = M_1 \times M_2$ along the leaves M_1 . Decisive is only that g has this “orthogonal-to- M_1 part” (the pullback of $g_{(2)}$) which is invariant under diffeomorphisms generated by Lie derivatives along the leaves M_1 (the invariance following precisely from the fact that it is a pullback). In fact, in (7) the component g_{11} , which corresponds to $g_{(1)}$

on $M_1 = \mathbb{R}$ in this picture, can depend on all the coordinates of M (and not just on x^1 as would be the case when coming from a pullback). Also in general we will not have to require a trivial fibration like in a Cartesian product; instead we will even permit singular foliations.

Let us confirm our expectation that in the case of $M = \mathbb{R} \times M_{(2)}$ equipped with the adapted coordinates (x^1, x^α) such that the metric tensor on M satisfies Equation (8), the gauge invariant content of (7) is indeed described by a sigma model of the type (1) with target $(M_{(2)}, g_{\alpha\beta})$. This is easy to see: Variation of (7) with respect to the gauge field A leads to $g_{11} * (dX^1 - A) = 0$, i.e. to $A = dX^1$. So the gauge field A is completely determined by means of the field $X^1 \in C^\infty(\Sigma)$. Since moreover X^1 is purely gauge according to (6), neither A nor X^1 contain any physical information. Varying, finally, the action (7) with respect to the remaining fields X^α , one obtains terms from the first line that give no contribution in the end due to $A = dX^1$, while the second line gives precisely the Euler Lagrange equations of the expected “reduced functional”.

In general, however, the physical degrees of freedom cannot be separated that easily from the unphysical ones. This is the main point of the use of gauge theories. In some sense they provide a smooth definition of an otherwise very singular quotient space. This is closely related to some ideas of non-commutative geometry or the resolutions of singular spaces in mathematics—but also to the use of Lie algebroids, as we will illustrate further below. So while in the simplest situations of clearly separable physical and gauge degrees of freedom the theory has to be constructed to give the expected results, the real interest lies in those situations where this separation is either hidden or not even possible (on a global level and in a smooth manner). We will now turn to the construction of in this sense more interesting gauge theories.

GENERALIZATION TO ARBITRARY FOLIATIONS

Consider the neighborhood of a point in M in which the leaves of the foliation are generated (over smooth functions) by a set of vector fields ρ_a , $a = 1, \dots, r$. Clearly they must be involutive, i.e. there will exist *functions* C_{ab}^c such that

$$[\rho_a, \rho_b] = C_{ab}^c \rho_c. \quad (10)$$

We want to promote *arbitrary* deformations along the leaves to a gauge symmetry, Equation (4) (with ε^a arbitrary functions on Σ , at least locally). The functional (7) corresponds to a special case of (3), only the transformation (9) deviates from the conventional transformation behavior. It is easy to verify that, as a consequence of involutivity, Equation (10), also in the more general situation where C_{ab}^c are some structure functions over

M (and not constant) the expressions (2) transform covariantly, $\delta(DX^i) = \varepsilon^a (\rho_a^i)_{,j} DX^j$ if Equation (5) holds true.[2] Thus, in generalization of (9), we make the ansatz

$$\delta A^a = d\varepsilon^a + C_{bc}^a A^b \varepsilon^c + \Delta A^a. \quad (11)$$

Here we wrote simply $(\rho_a^i)_{,j}$ and C_{bc}^a for $(\rho_a^i)_{,j}(X) \equiv X^*(\partial_i(\rho_a^i))$ and $C_{bc}^a(X) \equiv X^*(C_{bc}^a)$, respectively, and we will do likewise below. Variation of (3) with respect to (4) and (11) yields

$$\delta S = \int_\Sigma \frac{1}{2} \varepsilon^a (\mathcal{L}_{\rho_a} g)_{ij} DX^i \wedge *DX^j - g_{ij} \rho_a^i \Delta A^a \wedge *DX^j. \quad (12)$$

This vanishes for all $\varepsilon^a(\sigma)$ if and only if there exist some (X-dependent) coefficients ω_{bi}^a , corresponding to an $r \times r$ -matrix the coefficients of which $\omega_b^a \equiv \omega_{bi}^a dx^i$ are locally 1-forms on M , such that the following two equations hold true:

$$\Delta A^a := \omega_{bi}^a \varepsilon^b DX^i, \quad (13)$$

$$\mathcal{L}_{\rho_a} g = \omega_a^b \vee \iota_{\rho_b} g, \quad (14)$$

where \vee denotes the symmetric tensor product (for 1-forms $\alpha \vee \beta = \alpha \otimes \beta + \beta \otimes \alpha$). It is comforting to verify that this condition is independent of the chosen generators along the leaves. For example, using a change of generating vector fields,

$$\widehat{\rho}_a := L_a^b \rho_b, \quad (15)$$

where L_a^b are the components of an $r \times r$ matrix that are locally functions on M , this only changes the matrix ω : Indeed, (14) yields

$$\widehat{\omega}_a^c L_c^b = L_a^c \omega_c^b + dL_a^b, \quad (16)$$

which, in the case of an invertible matrix L reminds of the transformation property of a connection; we will come back to this observation in the final section below.

In fact, in the case of a regular foliation \mathcal{F} on a Riemannian manifold (M, g) there is a more geometrical formulation of the condition (14), which is presented in the first part of the following Theorem, the second part summarizing the main findings of this letter up to this point:

Theorem:

- Given a (regular) foliation \mathcal{F} of a Riemannian manifold (M, g) , then the orthogonal distribution $\mathcal{D} \perp T\mathcal{F}$ is invariant with respect to leaf-preserving diffeomorphisms on M , iff for any locally defined generating set of vector fields $(\rho_a)_{a=1}^r$ of $T\mathcal{F}$ there exists a local $r \times r$ matrix ω such that (14) holds true or, equivalently, iff

$$\mathcal{L}_v g \in \Gamma(T\mathcal{F} \vee TM). \quad (17)$$

- Let (M, g) be a pseudo-Riemannian manifold and \mathcal{F} a possibly singular foliation of M . Suppose that for any

local choice of $(\rho_a)_{a=1}^r$ and C_{ab}^c satisfying Eq. (10) there exists an ω_b^a such that Equation (14) holds true. Then the action functional (3) is gauge invariant with respect to the infinitesimal gauge transformations generated by $\delta X^i = \rho_a^i \varepsilon^a$ and $\delta A^a = d\varepsilon^a + C_{bc}^a A^b \varepsilon^c + \omega_{bi}^a \varepsilon^b (dX^i - \rho_c^i A^c)$.

Let us stress that in the first part of the theorem we did not write just foliation-preserving, but consider diffeomorphisms which preserve *each leaf* of the foliation separately. (To be on the safe side, one may want to exclude large diffeomorphisms of this kind, moreover; after all, we are considering only those diffeomorphisms generated by vector fields parallel to the given foliation).

Proof: Let us identify covariant and contravariant tensor fields on M by use of g . Given any symmetric 2-tensor, we can uniquely decompose it into three parts which belong to the space of sections of $S^2 T\mathcal{F} \equiv T\mathcal{F} \vee T\mathcal{F}$, $S^2 \mathcal{D}$, and $T\mathcal{F} \vee \mathcal{D}$, where $\mathcal{D} \equiv (T\mathcal{F})^\perp$ is the g -orthogonal complement to the tangent distribution of \mathcal{F} . Let g_\parallel and g_\perp be the first and the second components of g in this decomposition, respectively; note that by definition there is no third part of g , so $g = g_\parallel + g_\perp$. (In the very special case of \mathcal{F} defining a trivial fibration as discussed in the previous subsection, $g_\parallel = g_{(1)}$ and $g_\perp = g_{(2)}$). It is easy to verify that g is of the desired form, i.e. $\mathcal{L}_v g_\perp = 0$ for any (local) \mathcal{F} -tangent vector field v , if and only if the second component $(\mathcal{L}_v g)_\perp$ of $\mathcal{L}_v g$ is zero: Indeed, for any $v \in \Gamma(T\mathcal{F})$ and $v_1, v_2 \in \Gamma(\mathcal{D})$ one has $(\mathcal{L}_v g_\parallel)(v_1, v_2) = v(g_\parallel(v_1, v_2)) - g_\parallel([v, v_1], v_2) - g_\parallel(v_1, [v, v_2]) = 0$. Hence the second or parallel component of $\mathcal{L}_v g_\parallel$ is always zero, and thus the $S^2 \mathcal{D}$ -part of $\mathcal{L}_v g$ coincides with the one of $\mathcal{L}_v g_\perp$. Similarly one checks that $\mathcal{L}_v g_\perp \in \Gamma(S^2 \mathcal{D})$. We therefore conclude that

$$\mathcal{L}_v g_\perp = 0 \quad (18)$$

iff the second component of $\mathcal{L}_v g$ vanishes, i.e. in its decomposition there are only the first and third component remaining: Eq. (18) holds true iff $\mathcal{L}_v g \in \Gamma(S^2 T\mathcal{F}) \oplus \Gamma(T\mathcal{F} \vee \mathcal{D}) = \Gamma(T\mathcal{F} \vee TM)$. Obviously, (14) will guarantee the latter property. But also vice versa, \mathcal{F} being regular, one can always choose a smooth collection of such 1-forms ω_b^a under the condition (17). \square

Equation (18) is the appropriate searched-for generalization of the coordinate-dependent Equation (8). Its equivalent condition, Equation (17), gives a choice-independent formulation of Equation (14) for the case of a regular foliation, furthermore. Still, for the second part of the above Theorem, regularity of the foliation does not need to be assumed.

LIE GROUPOIDS VERSUS LIE GROUPS

Most of the investigations of Sophus Lie were concerned with transformations that infinitesimally satisfy conditions of the form (10). It was a highly non-trivial step to arrive from this to the abstract notion of a group and a Lie algebra. In the latter case it amounts to first restricting to cases where the structure functions C_{ab}^c in (10) can be chosen as constant, then postulating elements ξ_a having their proper own life: they are postulated to generate an algebra structure on the vector space \mathfrak{g} spanned by the ξ_a s by means of the product relation $\xi_a \bullet \xi_b = C_{ab}^c \xi_c$. [3]

Let us go back to the original setting of a (possibly singular) foliation with a local description such as in (10) and let us drop the somewhat unnatural condition, which underlie implicitly the introduction of Lie groups and algebras, that there should be a *global* choice of ρ_a s and C_{bc}^a such that the latter functions are *constants* over all of M . First, we do not require that the vector fields $(\rho_a)_{a=1}^r$ need to be defined everywhere; we content ourselves with the fact that any neighborhood of a point permits a set of such vector fields generating the given (singular) foliation, keeping, however, the number r fixed. On overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$ of local charts then certainly there will be $r \times r$ matrices L such that an equation of the type (15) holds true. If the ρ_a s form a basis at each point of a given neighborhood, then the matrices L are even unique (and satisfy automatically a cocycle condition). In general, however, the generating vector fields may be linearly dependent, and thus the transition matrices L not uniquely determined. Let us assume that also in this case there is a consistent choice of these matrices such that they are invertible on each overlap and that on triple overlaps they fit consistently.

Such a consistent choice of matrices L over an atlas of M is *equivalent* to the construction of a rank r vector bundle $E \rightarrow M$. In the very special case where the structure functions C_{bc}^a are also constant, one has $E = M \times \mathfrak{g}$ and one can separate the manifold M from the abstract vector space \mathfrak{g} . This is a very particular situation, however; even as a vector bundle, E may be far from trivializable.

How to obtain the algebraic structure on E that, in the above particular case, would reduce to the Lie algebra structure on \mathfrak{g} ? Since in general we cannot separate \mathfrak{g} from M inside E , let us thus look at $M \times \mathfrak{g}$ also in this case, where in principle we know the action ρ of \mathfrak{g} on M , $\rho: \mathfrak{g} \rightarrow \Gamma(TM)$, $\xi_a \mapsto \rho_a$. With the action of a Lie algebra on a manifold we are used to associating a nilpotent odd BRST-transformation in physics. The standard BRST-charge, known e.g. from ordinary Yang-Mills gauge theories in the infinite dimensional setting,

has, in the present coordinates, the simple form

$$Q = \xi^a \rho_a^i \frac{\partial}{\partial x^i} - \frac{1}{2} C_{bc}^a \xi^b \xi^c \frac{\partial}{\partial \xi^a}, \quad (19)$$

where the variables ξ^a are considered to be odd and, in the BRST-language or the more general BV-language are called “ghosts”. $Q^2 = 0$ then follows from the above data, but also permits reciprocally to *encode* the algebraic structure on \mathfrak{g} as well as its action on M . It is near at hand to consider the *same* odd vector field (19) also in the more generic case where the C_{bc}^a s are not constant. We now require that again Q squares to zero. (Let us remark in parenthesis that $Q^2(x^i) = 0$ follows already automatically from Eq. (10), to which it is equivalent in fact. Moreover, the contraction of the remaining condition $Q^2(\xi^a) = 0$ with ρ_a^i also follows already from that equation, by the use of the Jacobi identity of the commutator on the left-hand-side. The additional input at this stage seems very mild therefore.) It is a mathematical fact [4] (cf, e.g., [5] for more details) that a nilpotent vector field (19) equips the above vector bundle E with what is called the structure of a Lie algebroid!

As the name suggests, Lie algebroids are an infinitesimal version to Lie groupoids. The notion of Lie groupoids as a generalization of Lie groups is best understood intuitively by looking at the example of homotopy classes of paths on a manifold M (called the fundamental groupoid of M) and the homotopy classes of paths starting and ending at a given $p \in M$ (called the fundamental group of M with base point p). We refer to the mathematics literature [6–8] for definitions and known facts of Lie groupoids.

In contrast to ordinary Lie algebras, not any Lie algebroid integrates to a Lie groupoid. The precise conditions for the integration were found only rather recently [9]. The gauge invariant action functional (3) makes sense, however, also in the case of a non-integrable Lie algebroid action on M , such as the Poisson sigma model [10, 11] also makes perfectly sense with a target being a non-integrable Poisson manifold.

GENERALIZED GAUGE FIELDS AND EXTENDED KILLING EQUATION

One of the most pertinent physical question posing itself in the following is how to generate a “kinetic term” for the gauge fields A^a . After all, this is what finally leads to the observable interaction bosons in the conventional Standard Model. Here it is important to remark that in general we cannot separate \mathfrak{g} from M in the context of a Lie algebroid E ; thus, the scalar (matter) and the 1-form (gauge) fields should consequently be considered as coming together, generically in a non-separable way. This was also one of the lessons learnt from the toy sigma model mentioned in the first footnote, cf. [12, 13].

A combined concise and meaningful joint description of the collection of fields X^i and A^a was found to be provided by simply vector bundle morphisms from $T\Sigma$ to E . Thus, even in the case of a general Lie algebroid E effecting to gauge its orbits, the functional (3) is a functional $S[a]$ of such maps $a: T\Sigma \rightarrow E$.

In view of the modified transformation law of the gauge fields, which even in the 1-dimensional case takes the apparently intricate form (9), impeding to add simply the standard abelian Yang-Mills term as a kinetic part for the gauge field, one notices that this question is not completely obvious to answer in a physically satisfactory manner.

Returning to Equation (16), however, we now recognize that the geometrical significance of ω_b^a is the one of a *connection* on the bundle E . With this the gauge transformations found in the second part of our Theorem above are precisely one of the two options of gauge symmetries proposed from independent, more mathematical considerations in [12–14]! In [13], moreover, possible purely kinetic terms for Lie algebroid Yang-Mills theories were proposed and subsequently their coupling to matter fields considered in [14]. It will be interesting to further investigate such coupled systems from the present perspective of an ordinary sigma model (1), gauged in the more general framework proposed in the present paper in terms of $S[a]$. We stress, however, that even without a kinetic term, the much more general gauging principle is meaningful already by itself.

The observation that ω_b^a is nothing but a connection on E has also another interesting geometrical consequence, providing an alternative interpretation of the main equation (14) underlying the gauging. It is well-known that Killings equation $\mathcal{L}_v g = 0$ can be rewritten in the form

$$v_{i;j} + v_{j;i} = 0, \quad (20)$$

the indices of the vector field are lowered by means of the metric g and the semicolon indicates a covariant derivative with respect to the Levi-Civita connection of g . Any generating vector v of a symmetry of g has to satisfy this equation.

We may reformulate the condition we found from gauging in a similar way: It is a condition on a collection of r vector fields $v_a \equiv \rho_a$ satisfying the condition (10) and (14). Locally we can always consider these set of vector fields v_a on M as a single section of $E^* \times TM$, where E is a trivial rank r bundle. In fact, in the previous section we assumed that there exists a consistent gluing over local charts so as to define a not necessarily trivial rank r bundle $E \rightarrow M$. The collection of vector fields v_a now corresponds to a *single* (at least locally defined) section $v \in \Gamma(E^* \otimes TM)$ where $v = v_a^i e^a \otimes \partial_i$ with e^a being a local basis in E^* . g permits v to be identified with a section of $E^* \otimes T^*M$ again, certainly. Using ω_b^a as a connection in E and the Levi-Civita connection on T^*M we arrive at

the following compact generalization of Equation (20):

$$(\nabla v)_{\text{symm}} = 0 \quad \Leftrightarrow \quad v_{ai;j} + v_{aj;i} = 0. \quad (21)$$

Certainly, an ordinary Lie algebra action of symmetries is a very particular case inside this: Then E is the globally flat bundle $E = M \times \mathfrak{g}$ (equipped with its canonical flat connection, $\omega_b^a = 0$ in an auto-parallel frame e^a) and the above extended Killing equation (21) reduces to the standard Killing equation (20) to be satisfied for each of the individual vector fields (or 1-forms) v_a separately.

CONCLUSION AND OUTLOOK

We have seen that the unbiased attempt to gauge a sigma model, not following blindly the established path of Lie groups acting as isometries on the target M , but giving it a fresh look similar to the original perspective of Sophus Lie, we are almost automatically led to the notion of Lie algebroids E and, if they exist, their Lie groupoids. And even if these are action Lie algebroids, i.e. $E = M \times \mathfrak{g}$ for some Lie algebra \mathfrak{g} , we do not need \mathfrak{g} to be an isometry for gauging, but only that the metric satisfies (17) or, if \mathcal{F} is singular, the more general condition (21) so as to gauge the (possibly singular) foliation \mathcal{F} generated by \mathfrak{g} .

Many interesting physical and mathematical questions pose themselves naturally in the present context, like those addressed in the previous section on the physical side or a further investigation of the extended Killing equation and its related Lie algebroid geometry on the mathematical side. We intend to come back to at least some of these questions in a more extended version of this paper elsewhere.

Lie algebras and Lie groups are a highly developed and important subject of mathematics and its use within classical and quantum physics undoubtedly indispensable. In the context of gauge theories, we find, however, that it seems much more natural to consider—at least also—Lie algebroids and Lie groupoids. This all the more so due to the considerable progress in the recent mathematics literature on this subject. Nature is known to have made ample use of the notion of Lie groups. It seems unlikely to our mind that Nature has restricted itself to this relatively rigid notion, not also making use of the much more flexible one of Lie groupoids—and this possibly also at the level of fundamental physics. To be unravelled.

- [1] One may wonder why then this was not observed earlier. But although close in spirit to Lie's original work about "local symmetries", Lie groupoids have received appropriate attention even inside mathematics only within about the last two decades. This in turn is partially due to substantial recent progress in the context of the solution of two old major mathematical problems, the one of deformation quantization [15] as well as the one of integration of Lie algebroids to Lie groupoids [9]. Interestingly, both these advances were triggered and can be understood [16, 17] by means of a sigma model appearing first in the physics literature [10, 11] two decades ago. We add a remark that K. Gawedzki shared with us: "The mathematical notion of a (non-abelian) Lie group is in fact not so much simpler than the one of a Lie groupoid; only within physics we became already used to Lie groups while Lie groupoids appear still exotic to us. It is near at hand that this will change at the end of the day." We hope to contribute to this by means of the present work.
- [2] We remark that C_{bc}^a is not yet uniquely defined by means of (10) since we did not demand the set of ρ_a s to form a basis. We will address this ambiguity in the subsequent section. At the moment, we assume that any smooth choice consistent with Eq. (10) has been made.
- [3] Conventionally the bullet \bullet is denoted by a bracket $[\cdot, \cdot]$ certainly—we wanted to make aware of the fact that a priori this abstract product has not much to do with the commutator bracket of the generators (10); that finally this product can be reinterpreted as a commutator of left-invariant vector fields on a group manifold "integrating" this Lie algebra is a highly non-evident fact at this point, demanding to develop the theory much further.
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